Undecidability and temporal logic: some landmarks from Turing to the present

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Outline

- Introduction: a brief retrospective.
- Turing's undecidability of the Halting Problem (HP) from temporal logics perspective.
- Undecidability of interval temporal logics by reduction from the HP
- Undecidability by reduction from tiling problems
- Undecidability of temporalized logics
- Undecidability of quantitative temporal logics
- Outlook and concluding remarks

The focus of this talk

- Undecidability of the satisfiability/validity problem;
  (almost) no model checking;
- Only propositional temporal (and modal) logics;
  (almost) no first-order logics;
- Details of some interesting cases and an overview of the rest.

  No claim of completeness!

Introduction: a brief retrospective

- 1936: Turing proves the undecidability of the Halting problem.
- 1936: Church proves the undecidability of first-order logic.
- 1957-1967: Prior introduces formal temporal logic.
- 1962: Büchi: decidability of the MSO of one successors
  1969: Rabin: decidability of the MSO of two successors
- 1974: Burstall proposes the use of temporal logics in CS.
  1977: Pnueli introduces LTL and proposes the use of temporal logics to specification and verification of reactive systems.
- Since early 1970s: many decidability results for propositional modal and temporal logics, using FMP or Büchi/Rabin results. Modal/temporal logics praised for their "robust decidability".
- However, since the mid 1970s a variety of undecidability results in propositional temporal logics emerges, too.
- In retrospective: Turing's undecidability of the Halting problem is the first such undecidability result.

Turing's undecidability of the Halting problem from temporal logic perspective

The halting of a Turing machine as a temporal logic formula

- The configuration graph of a TM as a transition system $\color{red}{Conf(M)}$:
  - States: configurations of the TM
  - Transitions: determined by the TM transition relation
  - Labels for initial and terminal states
- Temporal logic for Turing machines:
  - Atomic propositions `init` for the initial states and `term` for terminal states, plus temporal operators incl. $\mathcal{X}$ and $\mathcal{F}$.
  - Expressing the halting property (for deterministic TM) :

$$\texttt{init} \rightarrow \mathcal{F}\,\texttt{term}$$

The Halting problem as a model checking problem

The Halting problem as a local model checking problem:

- The halting of a Turing machine M on any given input is equivalent to the truth of $\mathcal{F}\,\texttt{term}$ at the corresponding initial state in $Conf(M)$.
- Thus, the undecidability of the Halting problem translates into an undecidable local model checking problem in the class of transition systems of type $Conf(M)$.
- Applying this to the universal Turing machine U yields an undecidable local model checking problem on $Conf(U)$ alone.

The Halting problem as a global model checking problem:

- The halting of M on every given input is equivalent to the validity of $\texttt{init} \rightarrow \mathcal{F}\,\texttt{term}$ in $Conf(M)$.
- The undecidability of the Halting problem implies that the problem whether a given TM always halts is undecidable.

The Halting problem as a validity problem

- Any Turing machine M can be described by a temporal logic formula $\Phi(M)$ in a *sufficiently expressive* temporal language.
- The Halting problem for M on a blank tape is equivalent to the validity of

$$(\Phi(M) \wedge \texttt{init-blank}) \rightarrow \mathcal{F}\,\texttt{term}$$

Early undecidability results in propositional temporal logics

The first known to me undecidability result for a propositional temporal logic:

Steve Thomason
Reduction of Second-Order Logic to Modal Logic
Mathematical Logic Quarterly, vol 21, 1975, pp. 107-114

Reduction of the frame validity based logical consequence the MSO theory of a binary relation to a propositional tense logic T_{15} with a set of Prior's tense operators H_i, G_i, P_i, F_i over each of 15 temporal orderings $\preceq_1, \ldots, \preceq_{15}$, satisfying special interrelations.

Further, the logical consequence in T_{15} is reduced to logical consequence in plain modal logic.

NB: the reduction adds a special formula δ to the premises, so it does not reduce validity to validity.

Early undecidability results in propositional temporal logics cont'd

Stephen Isard
A Finitely Axiomatizable Undecidable Extension of K.
Theoria 43 (3), 1977, pp. 195-202.

This is seemingly the first undecidability result in modal logic, using reduction from the Halting problem (of Minsky machines).

Valentin Shehtman
Undecidable Propositional Calculi.
Problems of Cybernetics. Non-classical logics and their applications, vol. 75, 1982, p.74-116 (in Russian)

These results refer to specially constructed, mostly artificial, logics.

The first undecidability results on 'natural', purely temporal logics?

The answer seems to lead to interval logics.

Undecidability of interval temporal logics

Moszkowski's Propositional Interval Temporal Logic (PITL)

Joseph Halpern, Zohar Manna, and Ben Moszkowski
A Hardware Semantics Based on Temporal Intervals.
Tech Report STAN-CS-83-963, Stanford University, 1983.

Ben Moszkowski
Reasoning about Digital Circuits.
PhD Thesis, Stanford University, 1983.

PITL-formulae:

$$\phi ::= p \mid \neg\phi \mid \phi \wedge \psi \mid \bigcirc \phi \mid \phi ; \psi.$$

Models of PITL: based on (finite) discrete linear orderings.

Formulae are evaluated on discrete intervals:
finite sequences of *states* $\sigma = s_0, s_1, \ldots, s_n$, with $n \geq 0$.

PITL with locality: semantics and decidability

- Atomic propositions evaluated at states.
- Locality principle: the value of an atomic proposition over an interval is its value at the initial state of that interval.
- Semantics of 'next' operator $\bigcirc$
  $s_0, s_1, \ldots, s_n \models \bigcirc\varphi$, where $n > 0$, iff $s_1, \ldots, s_n \models \varphi$
- Semantics of 'chop' operator ;
  $s_0, s_1, \ldots, s_n \models \phi;\psi$ iff there is i where $0 \leq i \leq n$, such that $s_0, s_1, \ldots, s_i \models \phi$ and $s_{i+1}, \ldots, s_n \models \psi$.

THEOREM[Halpern and Moszkowski, 1983]: The satisfiability problem for PITL with locality is decidable, though [Kozen'92] with nonelementary complexity.

PITL: undecidability

THEOREM[Halpern and Moszkowski, 1983]: The satisfiability problem for PITL without locality is undecidable.

Proof idea: follows early 1980's work by Chandra, Halpern, Meyer and Parikh on process logics.

Uses undecidability of emptiness of the intersection of the languages of two context-free grammars (in Greibach normal form).

Given two context-free grammars $G1$ and $G2$, one can construct a PITL formula that is satisfiable iff the intersection of the languages generated by $G1$ and $G2$ is nonempty.

Proof details in:

Ben Moszkowski

A Hierarchical Completeness Proof for Propositional Interval Temporal Logic with Finite Time.

Journal of Applied Non-Classical Logics, Special issue on Interval Temporal Logics and Duration Calculi, vol. 14(12), 2004, pp 55–104

Undecidability of duration calculi

Duration calculus: extension of the PITL framework with the notion of a state and state duration.

Zhou Chaochen, C. A. R. Hoare and A.R. Ravn
A Calculus of Durations
Information Processing Letters, 40(5):269-276, 1991

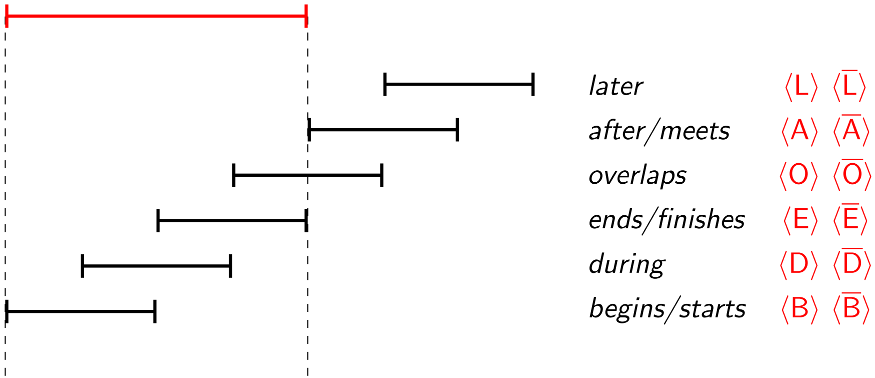
Even very simple fragments of DC are undecidable:

Michael R. Hansen and Zhou Chaochen
Duration Calculus: Logical Foundations
Formal Aspects of Computing, vol. 9 (1997), pp.283 330

Proof technique: reduction from the halting problem for 2-counter Minsky machines.

Allen's interval relations

J. F. Allen
Maintaining knowledge about temporal intervals.
Communications of the ACM, volume 26(11), pages 832-843, 1983.



Halpern-Shoham's interval logic

J. Halpern and Y. Shoham
A propositional modal logic of time intervals.
Proc. of LICS'1986, pp. 279-292.

J. Halpern and Y. Shoham
A propositional modal logic of time intervals.
Journal of the ACM, vol. 38(4), 1991, pp 935-962.

HS: a multimodal logic with modal operators associated with Allen's interval relations.

In the case of *non-strict semantics* when point intervals are allowed, it suffices to choose as primitive the modalities $\langle B\rangle, \langle E\rangle, \langle \overline{B}\rangle, \langle \overline{E}\rangle$ corresponding to the relations *begins, ends*, and their inverses:

$$\phi ::= p \mid \neg\phi \mid \phi \wedge \psi \mid \langle B\rangle\phi \mid \langle E\rangle\phi \mid \langle \overline{B}\rangle\phi \mid \langle \overline{E}\rangle\phi.$$

In the case of *strict semantics* without point intervals, the right and left neighbourhood modalities $\langle \mathrm{A}\rangle$ and $\langle \overline{\mathrm{A}}\rangle$ must be added.

Undecidability in interval logics: the bad news

Hereafter we assume the non-strict semantics, but all results apply to the strict semantics, too.

THEOREM[Halpern and Shoham'91]
The validity in HS over any class of ordered structures containing at least one with an infinitely ascending sequence is r.e.-hard.

Thus, in particular, HS is undecidable over the classes of all (non-strict) models, all linear models, all discrete linear models, all dense linear models, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, etc.

Proof idea: reduction from the *non-halting problem for Turing machines* to testing satisfiability in HS.

Undecidability in interval logics: can be worse...

THEOREM[Halpern and Shoham] The validity in HS over any class of Dedekind complete ordered structures containing at least one with an infinitely ascending sequence is Π^1_1-hard.

In particular, the validity in HS over any of the orderings of the natural numbers, integers, or reals is not recursively axiomatizable.

Proof: by reduction to satisfiability in HS of the *recurrence problem for non-deterministic TM*, asking for existence of a computation of a given NTM entering the start state infinitely often.

… and even worse

Undecidability occurs even without existence of infinitely ascending sequences. A class of ordered structures has unboundedly ascending sequences if for every n there is a structure in the class with an ascending sequence of length at least n.

THEOREM[Halpern and Shoham] The validity problem in HS interpreted over any class of Dedekind complete ordered structures having unboundedly ascending sequences is co-r.e. hard.

In particular, satisfiability of HS formulae in the finite is r.e. hard.

Proof idea: reduction from the halting problem for Turing machines to testing satisfiability in HS.

Some details of Halpern-Shoham's reduction setting the stage

Fix a Turing machine $M = \langle \{0,1\}, Q, q_0, q_f, \delta \rangle$.

Atomic propositions: $L = \{0, 1, *, \#, (q,0), (q,1), (q,B) : q \in Q\}$,

Truth in all future intervals: $[F]\phi := [A]\phi \wedge [L]\phi$.

A special propositional constant π, true at all point intervals.

Truth at the beginning/end of the current interval:

$$[[BP]]\phi := [B](\pi \rightarrow \phi); \quad [[EP]]\phi := [E](\pi \rightarrow \phi).$$

Every cell on the tape represented by an interval satisfying

$$\texttt{cell}(p) := [[BP]]\# \wedge [[EP]]\# \wedge [D]p \wedge \langle D \rangle p.$$

Some details of Halpern-Shoham's reduction IDs and configurations

ID: a sequence of cells, represented by an interval satisfying

$$\texttt{ID} := \langle B\rangle \texttt{cell}(*) \wedge \langle E\rangle \texttt{cell}(*) \wedge \langle D\rangle \bigvee_{l \in L, l \neq \#} \texttt{cell}(l) \wedge \neg \langle D\rangle \texttt{cell}(*)$$

Starting configurations:

$$\texttt{startID} := \texttt{ID} \wedge \langle D\rangle (\texttt{cell}((q_0, 0)) \vee \texttt{cell}((q_0, 1)) \vee \texttt{cell}((q_0, b))).$$

Final configuration:

$$\texttt{finalID} := \texttt{ID} \wedge \langle D\rangle (\texttt{cell}((q_f, 0)) \vee \texttt{cell}((q_f, 1)) \vee \texttt{cell}((q_f, b))).$$

Some details of Halpern-Shoham's reduction encoding computations

Computations of M are encoded as sequences of configurations:

$$*ID1**ID2**ID3**\ldots$$

To ensure matching the transition relation δ, a special atomic proposition `corr` is used, saying that an interval start and ends with cells that are corresponding in two consecutive IDs.

Describing `corr` is the most ingenious part of the reduction.

In the long run, the formula `computation` is defined, which is true of an interval iff it encodes a legitimate computation of M.

Now, non-halting is expressed by

$$\texttt{NoHalt} := \texttt{computation} \wedge [F]\neg\ \texttt{finalID}.$$

Hence, the reduction from non-halting of M to SAT(`NoHalt`).

For the satisfiability of `NoHalt`, any interval structure with an infinite ascending chain suffices.

Reduction from the halting problem

Note, that halting cannot be expressed by

$$\texttt{computation} \wedge F\ \texttt{finalID}.$$

because there may be non-standard models, e.g. on dense orders.

Such non-standard models can be eliminated on Dedekind complete orders by using the formula

$$\texttt{NoTelescope} := \neg\langle B\rangle[E]\langle D\rangle\texttt{cell}.$$

Eventually, the halting problem for M is reduced to satisfiability of

$$\texttt{Halt} := \texttt{computation} \wedge \texttt{standard} \wedge \langle B\rangle\texttt{startID} \wedge \langle E\rangle\texttt{finalID},$$

where `standard` is a formula ensuring that any interval starting and ending with IDs can be subdivided into a finite number of IDs.

On Dedekind complete structures one can also express the property of a computation to visit infinitely often its starting state.

Fragments of Halpern-Shoham's interval logic

Every subset of the 12 Allen's relations (excl. the equality) defines a fragment of HS.

Thus 4096 fragments arise; of them over 1000 expressively distinct.

([D. Della Monica, A. Montanari, VG., G. Sciavicco, IJCAI' 2011]: in strict semantics there are 1347 expressively distinct fragments.)

We denote fragments by listing the letters representing the occurring modalities, e.g. BE, $OA\overline{A}$, etc.

Sharpening the undecidability: early results

An inspection of the formulae in the constriction shows that any of the fragments ABE and $\overline{A}BE$ suffices for these reductions.

By refining Halpern and Shoham's reduction, Lodaya proved in

Kamal Lodaya
Sharpening the Undecidability of Interval Temporal Logic.
ASIAN 2000, volume 1961 of LNCS, pages 290-298. Springer, 2000.

the following:

THEOREM The BE-fragment of HS is undecidable over the classes of dense linear interval structures, and consequently, over all linear interval structures.

COROLLARY The interval logic with 'Chop' alone is undecidable over the classes of all (dense) linear interval structures.

Undecidability of temporal logics via tiling

The Integer Grid Tiling Problem (IGTP)

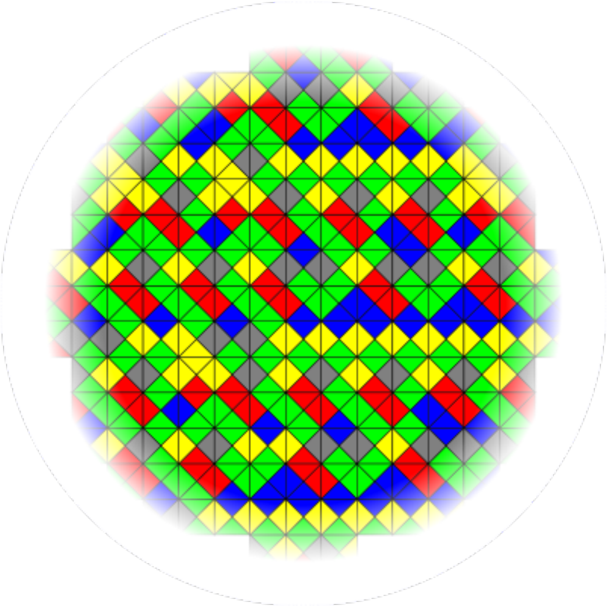
Tile: a 'square' with coloured sides: $\langle c_{up}, c_{right}, c_{down}, c_{left} \rangle$.

The $\mathbb{N} \times \mathbb{N}$ - tiling Problem:
Given a finite set of tile types $\mathcal{T} = \{t_1, \ldots, t_k\}$ of unlimited supply, can it be applied to tile the integer plane $\mathbb{N} \times \mathbb{N}$ by matching the respective colors of adjacent tiles?

THEOREM[Berger, 1966]
The Integer Grid Tiling Problem is undecidable.

The reason: there exist sets of tiles that can only tile the plane aperiodically.

Aperiodic tiling: example



Applications of tiling problems to logical undecidability/complexity

David Harel
Recurring Dominoes: Making the Highly Undecidable Highly Understandable
Ann. Disc. Math. 24 (1985), 51-72.

Reduction from the IGTP can be used to prove plain undecidability, i.e. non-recursiveness, but recursive enumerability.

Tiling can also be used to prove Σ_1^1-hardness, by reduction from the *recurrent Tiling problem*, asking for existence of tiling in which a given tile occurs infinitely often in the first row.

There are many decidable tiling problems. Polynomial reduction to them can be used to prove complexity results.

Generic proof of undecidability via tiling

Proving undecidability via reduction from the IGTP of a logic L:

1. Construct a formula GRID in L setting the grid.
2. Construct formulae in L describing the tiles in a given tile set.
3. Construct a formula in L describing correct tiling.
4. Translate any tiling problem to satisfiability of a formula of L.

Early undecidability via tiling results in temporal logic

Sample results in [Harel'85]: satisfiability in each of the following is Σ_1^1-hard by reduction from the recurrent Tiling problem:

- Quantified LTL(X,F),
- 2-dimensional LTL(X,F),
- the temporal spatial logic combining LTL(X,F) with K4.

Edith Spaan
Complexity of modal logics
Ph. D. Thesis. University of Amsterdam, 1993

Proves undecidability and Σ_1^1-hardness via tiling of the satisfiability of various modal logics; in particular, logics obtained from decidable ones by extending with universal modality.

An easy proof of undecidability via tiling

Consider the two-dimensional temporal logic X^2 with "next" operators for each of the 4 directions $\langle\uparrow\rangle$, $\langle\rightarrow\rangle$, $\langle\downarrow\rangle$, $\langle\leftarrow\rangle$, as well as a global modality, interpreted over the integer grid.
Then there is a straightforward encoding of IGTP, following

Mark Reynolds
Two-dimensional temporal logic
Proc. of Logic Colloquium '96, Springer-Verlag, 1998, pp. 219-236.

1. The formula GRID is not necessary, only needed to indicate the origin, by $\neg\langle\leftarrow\rangle\top \wedge \neg\langle\downarrow\rangle\top$.
2. Every tile τ is treated as an atomic proposition.
3. The formula describing correct tiling is:

$$[U]\left(\bigvee_{\tau\in T}\tau \wedge \bigwedge_{\tau\neq\tau'}\neg(\tau\wedge\tau') \wedge \bigwedge_{up(\tau)\neq down(\tau')}\neg(\tau\wedge\langle\uparrow\rangle\tau') \wedge \bigwedge_{right(\tau)\neq left(\tau')}\neg(\tau\wedge\langle\rightarrow\rangle\tau')\right)$$

4. The tiling problem readily translates into satisfiability of the conjunction of the above in $\mathbb{N}\times\mathbb{N}$.

A difficult proof of undecidability via tiling of the Compass Logic

Compass Logic [Venema'90]: a two-dimensional modal logic interpreted on products of two linear orders, with modal operators for each coordinate direction. NB: no 'next time' operators. Yet:

THEOREM[Marx and Reynolds'1997] The satisfiability in the compass logic is undecidable.

Maarten Marx and Mark Reynolds
Undecidability of the compass logic
Journal of Logic and Computation, 9(6), 1997, pp. 897-914.

Proof by elaborated encoding of the tiling problem.

NB: high undecidability on $\mathbb{N} \times \mathbb{N}$ was proved earlier by Spaan, by reduction from the recurrence problem for NTM, in:

Edith Spaan
Nexttime is not necessary
Proc of TARK'1990, 241-256

Undecidability of hybrid logic with binders via tiling

Digression: hybrid logics

Hybrid modal/temporal logics bring useful features of first-order logic into modal logic, thus boosting the expressiveness of ML without affecting its good computational properties.

Historical origins: Prior and Bull, in tense logic.

Explicitly developed since the early 1980's.

Main hybrid logic features

- Nominals: referring to single worlds in the model.
  Intuition: time stamps, 'clock variables'. Formally:

  $$(W, R, V), u \models \mathrm{i} \;\text{ iff }\; V(\mathrm{i}) = u$$

- Universal/global modality: referring to all worlds in the model:

  $$M, u \models [\mathrm{U}]\phi \;\text{ iff }\; M, w \models \varphi \;\text{ for every } w \in M.$$

- Satisfaction operators: refer to the truth at a named world:

  $$(W, R, V), u \models @_{\mathrm{i}}\phi \;\text{ iff }\; (W, R, V), V(\mathrm{i}) \models \phi.$$

- State variables: like nominals, but with no fixed interpretation. Assigned values by a separate variable assignment and used for reference to earlier stored possible worlds.
- Reference pointers/binders: refer to the current world.
  $\downarrow_s \varphi$ means: 'φ *is true if the current world is assigned to* s'.

  Formally: $M, g, u \models \downarrow_s \varphi$ iff $M, g[s \leftarrow u], u \models \varphi$, where $g[s \leftarrow u]$ is the assignment g, modified by assigning u to $g(s)$.

Examples on the expressiveness with binders

- The difference modality is definable in $\mathcal{H}([\mathrm{U}], \downarrow)$:

$$[\mathrm{D}]\varphi = \downarrow_s [\mathrm{U}](\neg s \rightarrow \varphi).$$

- Nominals can be modelled $\mathcal{H}([\mathrm{U}], \downarrow)$:

$$NOM(\varphi) = \langle U \rangle(\varphi \wedge [\mathrm{D}]\neg\varphi)$$

- *Until* and *Since* are definable in the tense hybrid logic $\mathcal{H}_t(\downarrow)$:

$$p\mathcal{U}q = \downarrow_s (Fq \wedge (H(Ps \rightarrow p))),$$

  and likewise for Since.

- *Until* is definable even in $\mathcal{H}(@, \downarrow)$:

$$p\mathcal{U}q = \downarrow_s \Diamond \downarrow_t (q \wedge @_s\Box(\Diamond t \rightarrow p)).$$

Undecidability of $\mathcal{H}([\mathrm{U}], \downarrow)$ via tiling

THEOREM[G.,1994] The satisfiability in $\mathcal{H}([\mathrm{U}], \downarrow)$ is undecidable, by reduction from the Integer Grid Tiling Problem.

Valentin Goranko
Temporal Logic with Reference Pointers,
Proc. of ICTL'94, Springer, LNAI 827, 1994, pp. 133-148.

Later, strengthened [Areces, Blackburn, and Marx, 1999] to undecidability of $\mathcal{H}(@, \downarrow)$.

The encoding is not straightforward, but is quite intuitive:

> *the formula GRID(p, q) says that every point of the model has exactly two successors: at one of them the value of p changes and the value of q remains the same (the move "to the right"), while at the other (the move "upwards") the opposite happens. Moreover, the routes "right;up" and "up;right" converge.*

Setting the Grid in $\mathcal{H}([\mathrm{U}], \downarrow)$

$$\begin{aligned}
\varphi_1 \quad = \quad & [\mathrm{U}]((p \wedge q \rightarrow F(p \wedge \neg q) \wedge F(\neg p \wedge q) \wedge G((p \wedge \neg q) \vee (\neg p \wedge q))) \;\wedge \\
& (p \wedge \neg q \rightarrow F(p \wedge q) \wedge F(\neg p \wedge \neg q) \wedge G((p \wedge q) \vee (\neg p \wedge \neg q))) \;\wedge \\
& (\neg p \wedge q \rightarrow F(\neg p \wedge \neg q) \wedge F(p \wedge q) \wedge G((\neg p \wedge \neg q) \vee (p \wedge q))) \;\wedge \\
& (\neg p \wedge \neg q \rightarrow F(\neg p \wedge q) \wedge F(p \wedge \neg q) \wedge G((\neg p \wedge q) \vee (p \wedge \neg q)))),
\end{aligned}$$

$$\begin{aligned}
\varphi_2 \quad = \quad & [\mathrm{U}] \downarrow_s ((p \wedge q \rightarrow [\mathrm{U}](Fs \rightarrow G(p \wedge q \rightarrow s))) \;\wedge \\
& (p \wedge \neg q \rightarrow [\mathrm{U}](Fs \rightarrow G(p \wedge \neg q \rightarrow s))) \;\wedge \\
& (\neg p \wedge q \rightarrow [\mathrm{U}](Fs \rightarrow G(\neg p \wedge q \rightarrow s))) \;\wedge \\
& (\neg p \wedge \neg q \rightarrow [\mathrm{U}](Fs \rightarrow G(\neg p \wedge \neg q \rightarrow s)))),
\end{aligned}$$

$$\begin{aligned}
\varphi_3 \quad = \quad & [\mathrm{U}] \downarrow_s ((p \wedge q \rightarrow [\mathrm{U}]((\neg p \wedge \neg q \wedge FFs) \rightarrow GG(p \wedge q \rightarrow s))) \;\wedge \\
& (p \wedge \neg q \rightarrow [\mathrm{U}]((\neg p \wedge q \wedge FFs) \rightarrow GG(p \wedge \neg q \rightarrow s))) \;\wedge \\
& (\neg p \wedge q \rightarrow [\mathrm{U}]((p \wedge \neg q \wedge FFs) \rightarrow GG(\neg p \wedge q \rightarrow s))) \;\wedge \\
& (\neg p \wedge \neg q \rightarrow [\mathrm{U}]((p \wedge q \wedge FFs) \rightarrow GG(\neg p \wedge \neg q \rightarrow s)))).
\end{aligned}$$

$$GRID(p, q) = p \wedge q \wedge \varphi_1 \wedge \varphi_2 \wedge \varphi_3$$

Describing the tiles in PL

Consider a tiling problem with a set of tiles $T = \{t_1, ..., t_m\}$ with colours $C = \{c_1, ..., c_k\}$.

Every tile has four sides: "up", "down", "left" and "right", each coloured in one of the colours from C.

To every colour c_i we assign four propositional variables u_i ("up"), d_i ("down"), l_i ("left"), and r_i ("right").

Each tile t with sides "up", "down", "left" and "right" coloured respectively in $c_{i_1}, c_{i_2}, c_{i_3}$, and c_{i_4}, we represent by the formula

$$\theta_t = (u_{i_1} \wedge \bigwedge_{j \neq i_1} \neg u_j) \wedge (d_{i_2} \wedge \bigwedge_{j \neq i_2} \neg d_j) \wedge (l_{i_3} \wedge \bigwedge_{j \neq i_3} \neg l_j) \wedge (r_{i_4} \wedge \bigwedge_{j \neq i_4} \neg r_j).$$

Describing the tiling in $\mathcal{H}([\mathrm{U}],\mathrm{G})$

Now we define the formulae:

$$COVER_T = [\mathrm{U}](\bigvee_{i=1}^{m} \theta_i)$$

which says that the model is properly tiled, i.e. every point in the model is covered by exactly one tile.

$$\begin{aligned} MATCHUP \;=\; & [\mathrm{U}](\bigwedge_{i=1}^{k}(u_i \to (p \wedge q \to G(p \wedge \neg q \to d_i)) \wedge \\ & (p \wedge \neg q \to G(p \wedge q \to d_i)) \wedge \\ & (\neg p \wedge q \to G(\neg p \wedge \neg q \to d_i)) \wedge \\ & (\neg p \wedge \neg q \to G(\neg p \wedge q \to d_i)))), \end{aligned}$$

which says that the colour "up" of each tile of the cover matches the colour "down" of the one above it;

Likewise, MATCHRIGHT is defined.

Translating the tiling problem in $\mathcal{H}([\mathrm{U}], \downarrow)$

Finally, we put

$$\Phi_T := GRID \wedge COVER_T \wedge MATCHUP \wedge MATHCHRIGHT$$

THEOREM
Φ_T is satisfiable if and only if $\mathbb{N} \times \mathbb{N}$ can be properly tiled by T.

More undecidability via tiling

The undecidability result uses the relative strength of the language $\mathcal{H}([\mathrm{U}], \downarrow)$ but no special properties of the models.

A number of similar results were established in the 1990s.

For instance [Spaan, 1996]: "*there is a uni-modal, decidable, finitely axiomatizable, and canonical logic for which adding the universal modality causes undecidability and for which adding the reflexive transitive closure modality causes high undecidability.*"

Edith (Spaan) Hemaspaandra
The Price of Universality,
Notre Dame J. Formal Logic Volume 37, Number 2 (1996), 174-203.

See also Spaan's PhD thesis, as well as many more undecidable polymodal logics in:

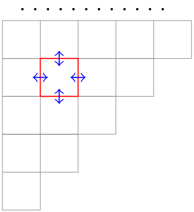
Marcus Kracht
Highway to the Danger Zone,
Journal of Logic and Computation, vol. 5(1996), pp. 93-109.

Undecidability of interval temporal logics by reduction from the Octant Tiling Problem

The Octant Tiling Problem

The 2nd octant of $\mathbb{Z} \times \mathbb{Z}$:

$$\mathcal{O} = \{(i,j) : i,j \in \mathbb{N} \wedge 0 \leq i \leq j\}$$



A natural interpretation of intervals on $\mathbb{N}$ into $\mathcal{O}$.

The Octant Tiling Problem: can a given finite set of tile types $\mathcal{T} = \{t_1, \ldots, t_k\}$ tile $\mathcal{O}$ while respecting the color constraints?

THEOREM The Octant Tiling Problem is undecidable.

Proof: by reduction from the tiling problem for $\mathbb{N} \times \mathbb{N}$, using König's Lemma.

Undecidability of interval logics via tiling: generic construction

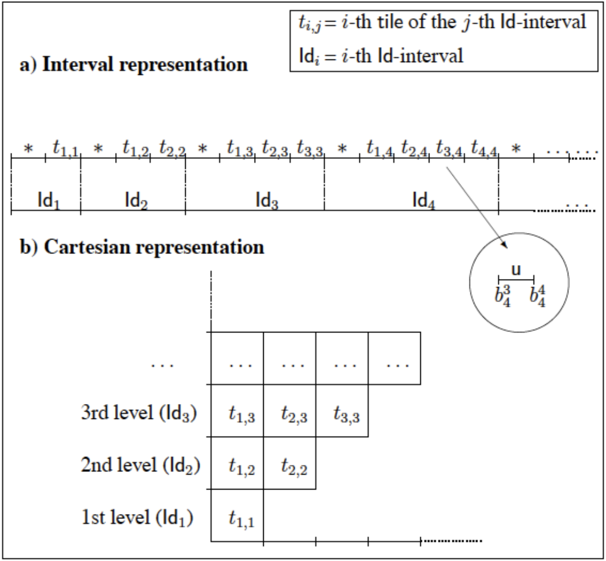
Given a finite set of tiles, we consider a signature containing, inter alia, special propositional letters u, tile, Id, $\mathtt{t_1, \dots, t_k}$, cbb, cbe, ceb, corr, and possibly others. The letters t_i represent the tiles.

The tiling framework is set by forcing the existence of a (usually unique) infinite chain of unit-intervals (u-*intervals*) on the linear order, which covers an initial segment of the interval model.

Unit intervals are used to place tiles and delimiting symbols.

Then, ID-intervals are introduced to represent the layers of tiles.

Undecidability of the interval logics via tiling: generic construction cont'd



Undecidability of the interval logics via tiling: generic construction cont'd

Each ID-interval must have the right number of tiles, and they must match horizontally: the Right-Neighbour relation.

The most challenging part usually is to ensure that the consecutive ID-intervals match vertically: the Above-Neightbour relation.

For that, we use several auxiliary propositional letters to refine and implement the idea of `corr`: cbb for matching the beginning point of a tile to the beginning point of the corresponding tile above; cbe, for matching beginning point with ending point above, and ceb for matching ending point with a beginning point above.

Undecidability of the interval logics via tiling: generic construction completed

Eventually, we encode the given Octant tiling problem by specifying the matching conditions between adjacent tiles.

The specific part of the construction is to use the given fragment of HS to set the chain of unit intervals and to express all necessary properties of IDs, the propositional letters for correspondence intervals, and the tile matching conditions.

For instance, using the After modality A the matching conditions can be expressed as follows, where $[F]p := [A]p \wedge [A][A]p$:

$$[F]((\mathrm{tile} \wedge \langle A\rangle \mathrm{tile}) \rightarrow \bigvee_{right(t_i)=left(t_j)} (\mathtt{t_i} \wedge \langle A\rangle \mathtt{t_j})),$$

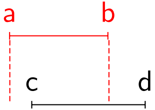
$$[F](\langle A\rangle \mathrm{tile} \rightarrow \bigvee_{up(t_i)=down(t_j)} (\langle A\rangle \mathtt{t_i} \wedge \langle A\rangle(\mathrm{cbb} \wedge \langle A\rangle \mathtt{t_j})))$$

A sample result using the Octant Tiling problem: undecidability of the logic O over (discrete) linear orderings

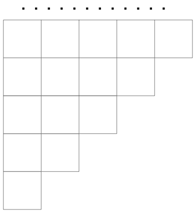
Semantics of the Overlap operator O:

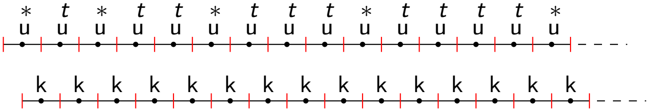
$\mathbf{M}, [a, b] \models \langle \mathrm{O} \rangle \phi$ iff

there exist c, d such that $a < c < b < d$ and $\mathbf{M}, [c, d] \Vdash \phi$.

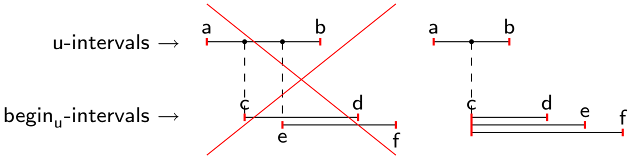


Encoding the Octant



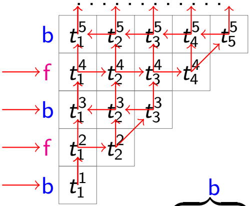


Encoding the Octant
u- and k-intervals of length 2



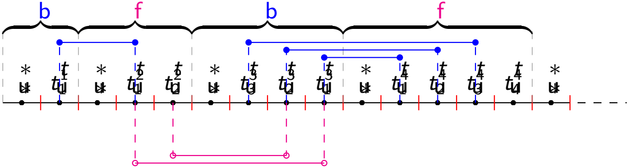
begin$_u$-intervals cannot overlap begin$_u$-intervals starting inside the same u-interval

Encoding the Above-Neighbour Relation



$$\text{up\_rel}^{\text{b}} \longrightarrow \neg\langle O\rangle \text{up\_rel}^{\text{b}}$$

$$\text{up\_rel}^{\text{f}} \longrightarrow \neg\langle O\rangle \text{up\_rel}^{\text{f}}$$



Undecidability of the logic O over discrete linear orderings

In the long run, for every finite set of tiles $\mathcal{T}$ we build a formula $\phi_{\mathcal{T}} \in$ O such that

$\phi_{\mathcal{T}}$ is satisfiable in a discrete linear ordering

iff

$\mathcal{T}$ can tile the 2nd octant.

THEOREM[Bresolin, Della Monica, G., Montanari, Sciavicco, 2009]
The satisfiability problem for the logic O is undecidable over any class of discrete linear orderings that contains at least one linear ordering with an infinite ascending sequence.

Likewise for $\overline{\text{O}}$, on classes having infinite descending sequences.

More recent results on undecidability of interval logics

Using variations of the Octant Tiling Problem encoding:

THEOREM The satisfiability problem for each of the HS fragments O, $\overline{\text{O}}$, AD, A$\overline{\text{D}}$, $\overline{\text{A}}$D, $\overline{\text{AD}}$, BE, B$\overline{\text{E}}$, $\overline{\text{B}}$E, $\overline{\text{BE}}$, is undecidable in any class of linear orders that contains at least one linear order with length greater than n, for each $n > 0$.

D. Bresolin, D. Della Monica, V. Goranko, A. Montanari, G. Sciavicco
The dark side of Interval Temporal Logic: sharpening the undecidability border
Proc. of TIME'2011

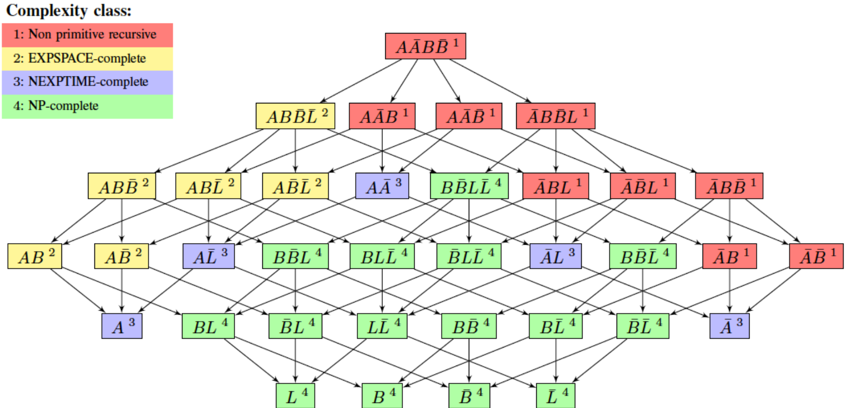
O and $\overline{\text{O}}$ were the first uni-modal fragments of HS proved undecidable over the class of discrete orderings.

A recent last blow by Marcinkowski and Michaliszyn (LICS'2011): undecidability of D and $\overline{D}$ over discrete linear orderings.

More recent results on undecidability of interval logics

D. Bresolin, D. Della Monica, A. Montanari, P. Sala, G. Sciavicco
Interval Temporal Logics over Finite Linear Orders: the Complete Picture
Proc. of ECAI'2012

Of the 1347 expressively different fragments of HS, only the following 35 and their symmetric versions are decidable over the class of finite linear orders:



Undecidability of temporalized logics

Temporalizing logics

Temporalization: combination of temporal and other logics, e.g.: products, fusions, etc.

Marcelo Finger and Dov Gabbay
Combining Temporal Logic Systems
Notre Dame Journal of Formal Logic 37(2): 204-232 (1996)

Temporalization often leads to undecidability.

Three important case studies:

- Products of modal and temporal logics
- Temporal epistemic logics
- Temporal description logics

Undecidability of products of logics

Dov Gabbay and Valentin Shehtman
Products of Modal Logics, Part I
L. J. of the IGPL, Vol. 6, No. 1, 1998, pp. 73-146

Frank Wolter
The decision problem for combined (modal) logics
Habilitationsschrift, Univ. of Leipzig, 1999

Dov Gabbay, Agi Kurucz, Frank Wolter, and Michael Zakharyaschev.
Many-dimensional modal logics: theory and applications
Elsevier, 2003.

Products of modal logics are massively undecidable. For instance:

- $ML(\mathbb{N} \times \mathbb{N})$ [Spaan, 1993]; $K[U] \times K[U]$ [Marx, 1999]; $K4.3 \times K4.3$ [Reynolds and Zakharyaschev 1999];
- Almost all three-dimensional modal/temporal logics. Related to the undecidability of FO^3.

Temporal-epistemic logics

Combine temporal and multi-agent epistemic logics.

An important earlier work with detailed proof of undecidability by reduction from the Halting Problem for Turing machines:

Richard Ladner and John Reif:
The Logic of Distributed Protocols
Proc of TARK'1986: 207-222

Various other developments during the 1980s. Unifying study in:

Joseph Halpern and Moshe Vardi
The complexity of reasoning about knowledge and time I: Lower bounds
Journal of Computer and System Sciences, 38(1), 1989, pp.195237

Joseph Halpern and Moshe Vardi
The complexity of reasoning about knowledge and time: Synchronous systems
IBM Research Report, 1989

A variety of temporal-epistemic logics

Semantics based on so called *interpreted systems*: sets of runs in a transition system with epistemic indistinguishability relations on the state space for each agents.

A variety of 96 logics, based on six parameters:

- number of agents (one or many),
- the language (with or without common knowledge, linear or branching time, etc.),
- recall abilities (no recall, bounded recall, perfect recall),
- learning abilities (learning or no learning),
- synchrony (synchronous or asynchronous),
- unique initial state.

Complexity of the validity in temporal-epistemic logics

	$CKL_{(m)}/CKB_{(m)}$, $m \geq 2$	$KL_{(m)}/KB_{(m)}$, $m \geq 2$	$KL_{(1)}/KB_{(1)}$
$\mathcal{C}_{(nf)}$, $\mathcal{C}_{(nf,sync)}$, $\mathcal{C}_{(nf,uis)}$, $\mathcal{C}_{(nf,sync,uis)}$	Π^1_1	nonelementary (time $ex(ad(\varphi)+1, c\lvert\varphi\rvert)$)	double-exponential time
$\mathcal{C}_{(nl)}$, $\mathcal{C}_{(nf,nl)}$, $\mathcal{C}_{(nf,nl,sync)}$, $\mathcal{C}_{(nl,sync)}$	Π^1_1	nonelementary (space $ex(ad(\varphi), c\lvert\varphi\rvert)$)	EXPSPACE
$\mathcal{C}_{(nf,nl,uis)}$	Π^1_1	Π^1_1	EXPSPACE
$\mathcal{C}_{(nl,uis)}$	co-r.e.	co-r.e.	EXPSPACE
$\mathcal{C}_{(nl,sync,uis)}$, $\mathcal{C}_{(nf,nl,sync,uis)}$	EXPSPACE	EXPSPACE	EXPSPACE
$\mathcal{C}$, $\mathcal{C}_{(sync)}$, $\mathcal{C}_{(sync,uis)}$, $\mathcal{C}_{(uis)}$	EXPTIME	PSPACE for $KL_{(m)}$, EXPTIME for $KB_{(m)}$	PSPACE for $KL_{(m)}$, EXPTIME for $KB_{(m)}$

Figure 1: The complexity of the validity problem for logics of knowledge and time

Both linear and branching time logics involving more than one agents become highly undecidable (Π^1_1-complete) under some combined assumptions, e.g., of both unbounded memory and common knowledge.

Sharpening the undecidability of temporal epistemic logics

Spaan showed that neither Nexttime nor Until are needed for most of these results, but the knowledge operator K and the temporal operator G suffice:

Edith Spaan
Nexttime is not necessary
Proc of TARK'1990, 241-256

Undecidability of temporal description logics

Description logics: very close to modal logics.
Involve concepts (unary predicates) and roles (binary predicates).

TBoxes: finite sets of concept inclusions.

Description logics can be temporalized in various ways:

Alessandro Artale and Enrico Franconi
A survey of temporal extensions of description logics,
Annals of Math. and Artificial Intelligence, vol. 30, 2000, pp.171–210.

Many undecidability consequences from Halpern-Shoham results.

Many more undecidability results for temporal description logics in:

Frank Wolter
The decision problem for combined (modal) logics
Habilitationsschrift, Univ. of Leipzig, 1999

Undecidability of temporal description logics, cont'd

More recent undecidability results for quite weak fragments in:

Carsten Lutz, Frank Wolter and Michael Zakharyaschev
Temporal Description Logics: A Survey,
Proc. of TIME'2008, pp.3-14

$\mathcal{ALC}$ is the basic propositionally closed description logic.

THEOREM Concept satisfiability in $LTL_{\mathcal{ALC}}$ w.r.t. T-Boxes and with a single rigid (over time) role is Σ^1_1-hard.

Proof: by reduction from the *recurrent tiling problem*: given a set of tile types T, decide whether it can tile $\mathbb{N} \times \mathbb{N}$ so that a given tile t appears infinitely often in the first row.

Also: concept satisfiability in $LTL_{\mathcal{SHIQ}}$ with rigid roles and without TBoxes is undecidable.

Proof: by reduction from Post's Correspondence Problem.

Undecidability of quantitative temporal logics

Real-time extensions of temporal logics

R. Alur and T. Henzinger
Logics and models of real-time: a survey
Real-Time: Theory in Practice, Proc. REX Workshop 1991, Springer, 1992, vol. 600 of LNCS, pp. 74106.

Real-time extensions of temporal logics:

- time-bounded operators:

$$G(p \rightarrow F_{=10} q)$$

- freeze quantification: very similar to hybrid binders: binds a variable x to the current time, e.g.:

$$Gx.(p \rightarrow Fy.(q \wedge y \leq x + 3)$$

- time variables and quantification, e.g.:

$$\forall x G(p \wedge T = x \rightarrow F(q \wedge T \leq x + 3)$$

Timed transition systems and timed automata.

Undecidability of metric temporal logics

Ron Koymans
Specifying real-time properties with metric temporal logic
Real-time Systems 2(4), 255299 (1990)

MTL augments the LTL operators with time bounded operators.

R. Alur and T. Henzinger
Real-time logics: Complexity and expressiveness
Information and Computation, 104(1):3577, 1993.

Punctuality causes undecidability: e.g., on discrete orderings with addition and on dense orderings with constant increment operation. Proof: by reduction from repeated reachability in Minsky machines.

A relaxed decidable version: MITL with interval constraints:

$$G(p \rightarrow F_{[2,10]} q)$$

J. Ouaknine and J. Worrell.
Some recent results in Metric Temporal Logic
Proc. of FORMATS'2008, LNCS 5215.

Undecidability of real-time logics

Timed propositional temporal logic TPTL: like LTL, but interpreted on time sequences and extended with freeze quantifiers.

R. Alur and T. Henzinger
A really temporal logic
Journal of the ACM 41:181-204, 1994.

Basic version decidable, but various extensions, e.g., time addition or multiplication by 2, or dense time domain cause Π^1_1-hardness.

Summary: what causes undecidability in temporal logics?

Propositional temporal logics are generally decidable, but adding some syntactic or semantic features can make them explode.

Many important types of temporal logics are generally undecidable, even under very weak assumptions.

What are the typical causes of undecidability in temporal logics?

- Grid-like models, many-dimensional or temporalized systems,
- Interval-based semantics, where truth of formulae is defined on time intervals, with no locality assumptions.
- Temporal operators along multiple (at least two) time-lines. Products of simple temporal logics.
- Time reference mechanisms, such as freeze quantifiers and hybrid binders.
- Arithmetic features: time addition, exact time constraints, etc.

Conclusion: is there life beyond decidability?

Yes, of course. Classical first-order logic is a witness.

Undecidability is bad, but how bad? And, (how) should we care?

Possible ways out:

- Syntactic restrictions, identifying decidable fragments (e.g. FO^2, guarded fragments, etc.)
- Suitable parametric restrictions, e.g. on number of propositional variables, depth of nesting, etc.

Maurice Margenstern
Frontier between decidability and undecidability: a survey
Theoretical Computer Science, vol. 231, no. 2, 2000, pp. 217-251

- Semantic restrictions, 'taming' the semantics. E.g.: locality.
- Semi-decision procedures, e.g. resolution or tableaux.
- Using heuristics and human-computer interaction, etc.

PostScriptum: Terminator vs Turing

Terminator: research project at Microsoft Research, Cambridge, focused on the development of automatic methods for proving program termination and general liveness properties.

`http://research.microsoft.com/en-us/um/cambridge/projects/terminator/`

See article *"Terminator Tackles an Impossible Task"*

`http://research.microsoft.com/en-us/news/features/terminator.aspx`

So, can the theoretically undecidable be practically decided?

The future will tell.

Maybe.

The end